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THERMOCONVECTIVE WAVES

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Аннотация—Предсказана возможность распространения слабозатухающих поперечных, тепловых и звуковых волн в вязкой теплопроводящей, термически сжимаемой жидкости при наличии гравитационного поля. Проанализирован спектр частот, глубина проникновения и длина волн, распространяющихся с малым затуханием. Обсуждается возможность создания механического генератора температурных волн и возбуждение поперечных и продольных колебаний тепловым способом.

NOMENCLATURE

x, y,	Cartesian coordinates;
u, v,	components of velocity along axes x, y, respectively;
T,	temperature of fluid;
ρ,	density of fluid;
p,	pressure;
λ,	thermal conductivity;
μ, ν,	viscosity and kinematic viscosity coefficients;
g,	gravitational acceleration;
ω,	frequency of oscillations;
k,	x-component of wave vector;
$V, U, \theta, \prod$,	amplitude of velocity, temperature and pressure fluctuations;
L,	length of penetration of thermoconvective waves;
l,	wavelength;
φ, ψ,	phases of thermoconvective waves;
A, B, a, b,	constants.

IT WILL be shown that in a viscous heat-conducting fluid, existing in a constant gravity field, weakly damped thermal and transverse waves may propagate. These problems are considered partly in [1, 2].

As is known [1–7], in an isothermal fluid transverse and thermal waves are strongly damped even in the absence of gravity forces. The amplitudes of these waves decrease by $\exp(2\pi) \simeq 540$ times per wavelength. Essentially, this is an aperiodic motion which it is more correct to call simply oscillatory rather than wave motion.

The case is different for a nonisothermal fluid. Assume that the temperature gradient exists in the direction of the gravity field. Under conditions of mechanical equilibrium the value of the gradient should be constant. If the direction of ∇T coincides with that of the gravity field, then a critical absolute value of the gradient exists above which convection arises [8–11]. It will be assumed that the conditions necessary for the presence of mechanical equilibrium are satisfied and it will be investigated how small perturbations of temperature, velocity, pressure and density propagate against the background of the above state of mechanical equilibrium with constant temperature gradient. A simple typical example is sufficient for elucidating the principal peculiarities of the process.

Assume that the fluid occupies the right-hand semi-space $x \geqslant 0$ and is bounded by a flat surface $x = 0$. The direction of the gravity field $\vec{g}$ is parallel but opposite in direction to the axis y. Let us consider that the axes x, y are perpendicular. There is a constant temperature gradient dT_0/dy along the axis y over the whole semi-space $x \geqslant 0$. Let the surface bounding the fluid oscillate along the axes x, y with a frequency ω and the surface temperature change with time by the harmonic law. It is required to determine the developed motion and distribution of temperature in the fluid. The above problem is a characteristic two-dimensional problem of simultaneous free and forced convection and is described by the following set of equations [4]

$$\rho\left[\frac{\partial\vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v}\right] v = -\nabla p + \mu\Delta\vec{V} + (\tfrac{1}{3}\mu + \mu_i)\nabla(\operatorname{div}\vec{v}) + \rho\vec{g}$$

$$\frac{\partial p}{\partial t} + \operatorname{div}(\rho\vec{v}) = 0$$

$$\rho c\left[\frac{\partial T}{\partial t} + (\vec{v}\nabla)T\right] = \lambda\Delta T - p\operatorname{div}\vec{v} + \mu\Phi$$

$$f(p, \rho, T) = 0 \qquad (1)$$

with the boundary conditions

$$x = 0 \qquad \begin{aligned} &T = T_0 + \theta_0\cos\omega t,\\ &\bar{v}_i = V_{0i}\cos(\omega t + \alpha_{1i}),\\ &\rho = \rho_0 + M_0\cos(\omega t + \alpha_2),\\ &p = p_0 + \textstyle\prod_0\cos(\alpha t + \alpha_3). \end{aligned} \qquad (2)$$

$$\lim_{x\to\infty}(T - T_0,\ v_i,\ \rho - \rho_0,\ p - p_0) = 0. \qquad (3)$$

Boundary condition (2) takes into account all possible periodic oscillations of density, pressure and all velocity components. If the oscillation amplitudes of the temperature T and velocity are small, the solution of equations (1)–(3) should be of the form

$$\begin{aligned} &T = T_0 + T'(x, t), \qquad v_i = v_i'(x, t),\\ &\rho = \rho_0 + \rho'(x, t), \qquad p = p_0 + p'(x, t). \end{aligned} \qquad (4)$$

The subscript zero indicates equilibrium distributions of temperature, density and pressure and T', p', ρ', their small perturbations. The velocity v_i' is also assumed small and of the same order of magnitude. By substituting expansions (4) into equations (1)–(3) and neglecting small values of order higher than the first, one gets the following linear system. (In further considerations the primes of the dimensionless values are omitted.)

$$\frac{\partial T}{\partial t} + \gamma v = \varkappa\frac{\partial^2 T}{\partial x^2}, \qquad (5)$$

$$\gamma = \frac{dT_0}{dy}\bigg/\left[1 - \frac{p_0}{\rho_0^2 C_{p0}}\left(\frac{d\rho}{dT}\right)_{T=T_0}\right],$$

$$\varkappa = \frac{\lambda}{\rho_0 C_{p0}}\bigg/\left[1 - \frac{p_0}{\rho_0^2 C_{p0}}\left(\frac{d\rho}{dT}\right)_{T=T_0}\right].$$

$$\frac{\partial v}{\partial t} = \nu\frac{\partial^2 v}{\partial x^2} - \beta T, \qquad (6)$$

$$\beta = \rho_0^{-1}g^y\left(\frac{d\rho}{dT}\right)_{T=T_0} = \text{const.}$$

$$\rho = T\left(\frac{d\rho}{dT}\right)_{T=T_0}. \qquad (7)$$

$$\frac{\partial\rho}{\partial t} + \rho_0\frac{\partial u}{\partial x} = 0. \qquad (8)$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \left(\frac{4}{3}\nu + \nu_1\right)\frac{\partial^2 u}{\partial x^2}. \qquad (9)$$

$$x = 0 \quad \begin{aligned} &T = \theta_0\exp(i\omega t), \quad v = V_0\exp(i\omega t),\\ &u = U_0\exp(i\omega t), \quad p = \textstyle\prod_0\exp(i\omega t). \end{aligned} \qquad (10)$$

$$\lim_{x\to\infty}(T, u, v, \varphi) = 0 \qquad (11)$$

where u, v are velocity components along the axes x and y, respectively. When deriving equations (5)–(9), density changes with pressure were neglected. Equilibrium distribution of density ρ_0 was assumed constant. This is evidently an unessential restriction at negative temperature gradient $dT_0/dy < 0$. In this case the change in density due to temperature compensates change in density due to pressure. In the case, when both fields act in the same direction, equations (5)–(9) are valid for a limited region along the y-axis.

Let us find periodic solution of equations (5)–(9) in the form of two-dimensional waves

$$\begin{aligned} T &= \theta \exp i(\omega t - kx), \\ v &= V \exp i(\omega t - kx), \\ u &= U \exp i(\omega t - kx), \\ p &= \prod \exp i(\omega t - kx). \end{aligned} \quad (12)$$

The first three equations may be integrated independently and lead to the following equation determing the relations between the wave vector k and frequency ω

$$k^4 \nu \varkappa + k^2 i\omega(\nu + \varkappa) - \omega^2 - \beta\gamma = 0. \quad (13)$$

To achieve damping of the waves at infinity, the condition is necessary that the wave vector be a complex one $k = k_1 + ik_2$, and its imaginary part negative $k_2 < 0$. The frequency is considered to be real. Then from (13) it is easy to obtain

(a) $$r = \omega^2(\nu - \varkappa)^2 - 4\varkappa\nu\beta\gamma \geqslant 0$$

$$k_1^2 = k_2^2 \quad (14)$$

$$k_2 = -\{A \pm (4\varkappa\nu)^{-1} r^{\frac{1}{2}}\}^{\frac{1}{2}}, \quad \beta\gamma > 0. \quad (15)$$

$$k_2 = -\{\pm A + (4\varkappa\nu)^{-1} r^{\frac{1}{2}}\}^{\frac{1}{2}},$$
$$\beta\gamma < 0, \quad |\beta\gamma| \geqslant \omega^2. \quad (16)$$

$$A = \frac{\omega(\nu + \varkappa)}{4\varkappa\nu}.$$

(b) $$r < 0 \qquad k_1 k_2 = -A, \quad (17)$$

$$k_2 = \{\mp B + (A^2 + B^2)^{\frac{1}{2}}\}^{\frac{1}{2}}, \quad (18)$$

$$B = (4\varkappa\nu)^{-1}(-r)^{\frac{1}{2}}. \quad (19)$$

From equations (14)–(18) it is seen that depending on the absolute value and sign of the term $\beta\gamma\omega^{-2}$ different types of thermal and transverse waves may exist in the fluid. For all the frequencies ω at $\beta\gamma < 0$ and $\omega \geqslant 2(\varkappa\nu\beta\gamma)^{\frac{1}{2}}/|\nu - \varkappa|$ at $\beta\gamma > 0$ only strongly damped waves (15), (16) exist, as in the case of an isothermal fluid in the absence of the gravity field. Indeed, the length of penetration (or damping) $L = |k_2^{-1}|$ i.e. is of the same order of magnitude as the wavelength $l = |2\pi k_1^{-1}|$. This case is of little interest.

The propagation of waves at frequencies $\omega < 2(\varkappa\nu\beta\gamma)^{\frac{1}{2}}/|\nu - \varkappa|$ in the medium with $\beta\gamma > 0$ is of great interest. In a fluid whose density decreases with increasing temperature $\beta < 0$ this means that the temperature gradient in the chosen system of coordinates should be negative, and vice versa. For gases the coefficient of thermal expansion is less than zero, hence the gravity field and temperature gradient should be parallel. In this case, as seen from (17), (18), two waves will exist of a given frequency ω. One corresponding to the plus sign at β in equation (18) is strongly damped. The other one corresponding to the component of the wave vector

$$\begin{aligned} k_1 &= A\{B[\surd(1 + A^2B^{-2}) - 1]\}^{-\frac{1}{2}} \\ k_2 &= -\{B[\surd(1 + A^2B^{-2}) - 1]\}^{\frac{1}{2}} \end{aligned} \quad (20)$$

can be weakly damped under certain conditions.

Let us ascertain these conditions. First, the absolute value of the damping length $L = |k_2^{-1}|$ should be large. This is possible when $A^2B^{-2} \ll 1$. Second, the penetration length measured in terms of wavelengths should also be large. To know when this condition is satisfied, the expression Ll^{-1} should be analyzed

$$\frac{L}{l} = \frac{k_1}{2\pi k_2} = \frac{A}{2\pi B[\surd(1 + A^2B^{-2}) - 1]}. \quad (21)$$

On simplifying equation (21) taking into account the condition $A^2B^{-2} \ll 1$

$$\frac{L}{l} \simeq \frac{[4\varkappa\nu\beta\gamma - \omega^2(\nu - \varkappa)^2]^{\frac{1}{2}}}{\pi\omega(\nu + \varkappa)}. \quad (22)$$

It is not difficult to see now that the ratio L/l will be the larger, the greater the coefficients of thermal expansion and the temperature gradient, the lower the frequency and the closer to each other the numerical values of the coefficients of kinematic viscosity and thermal diffusivity. As was already noted, the absolute value of the temperature gradient is limited, therefore there exists a finite upper limit of the frequencies $\omega < \sqrt{\beta\gamma}$ for the propagation of weakly damped waves. As the calculations show, low frequency oscillations are weakly damped. Let us give numerical estimations. The best media for the proposition at high frequencies are media for which $\nu \simeq \varkappa$. Such a medium is, for example, air. At temperature gradient of 10^2 °C/m and frequency of 10^{-3} c/s the penetration length is $\sim 10^2$ m or $\sim 10^4$ wavelengths. More detailed data are presented in Table 1.

Thus, is nonisothermal fluids with the temperature gradient parallel to the gravity field (or antiparallel if $\beta > 0$) weakly damped thermal and transverse waves may exist in the direction perpendicular to $\vec{g}$ and ∇T.

Amplitudes and phases of these waves depend upon conditions at the surface bounding the fluid. To satisfy boundary conditions (10) the solution has to be found as a sum of both weakly and strongly damped waves with frequency ω and wave vectors $k = k_1 + ik_2$, $\hat{k} = \hat{k}_1 + i\hat{k}_2$

$$T = Re\{\theta \exp i(\omega t - kx) + \hat{\theta} \exp i(\omega t - \hat{k}x)\}$$

$$v = Re\{V \exp i(\omega t - kx) + \hat{V} \exp i(\omega t - \hat{k}x)\}$$

where

$$\theta = \theta_1 + i\theta_2, \qquad \hat{\theta} = \hat{\theta}_1 + i\hat{\theta}_2,$$

$$V = V_1 + iV_2, \qquad \hat{V} = \hat{V}_1 + \hat{V}_2.$$

Without dwelling at length on rather a bulky calculation, write out the real parts of the expressions for the temperature and transverse velocity component

$$T = \theta_1 \cos^{-1}\varphi \exp(k_2 x)\cos(\omega t - k_1 x + \varphi) + \hat{\theta}_1 \cos^{-1}\hat{\varphi} \exp(\hat{k}_2 x)\cos(\omega t - \hat{k}_1 x + \hat{\varphi}). \quad (23)$$

$$v = (a\theta_1 - b\theta_2)\cos^{-1}\psi \exp(k_2 x)\cos(\omega t - k_1 x + \psi) + (a\hat{\theta}_1 + b\hat{\theta}_2)\cos^{-1}\hat{\psi} \exp(\hat{k}_2 x)\cos(\omega t - \hat{k}_1 x + \hat{\psi}) \quad (24)$$

where

$$\theta_1 = \frac{a\theta^0 + V_0\cos\alpha}{2a},$$

$$\theta^2 = \frac{V_0 \sin\alpha - b\theta_0}{2a}. \quad (25)$$

$$\hat{\theta}_1 = \frac{a\theta_0 - V_0\cos\alpha}{2a}, \qquad \hat{\theta}_2 = -\theta_2. \quad (26)$$

$$\varphi = \tan^{-1}\theta_2\theta_1^{-1}, \qquad \hat{\varphi} = \tan^{-1}\hat{\theta}_2\hat{\theta}_1^{-1} \quad (27)$$

$$\varphi = \tan^{-1}\frac{a\theta_2 + b\theta_1}{a\theta_1 - b\theta_2},$$

$$\hat{\varphi} = \tan^{-1}\frac{a\hat{\theta}_2 - b\hat{\theta}_1}{a\hat{\theta}_1 + b\hat{\theta}_2} \quad (28)$$

$$a = -\frac{\varkappa(-r)^{\frac{1}{2}}}{2\varkappa\nu\gamma}, \quad b = \omega\frac{\varkappa - \nu}{2\gamma\nu}. \quad (29)$$

The procedure is as follows:
Analyze the expressions obtained. First find out whether only one weakly damped wave can exist and if this is the case, what are the boundary conditions for it. It appears possible. Let the amplitude of transverse oscillations at the wall ($x = 0$) be $v_0 = a\,\theta_0 \cos^{-1}\alpha$, and the phase shift between temperature and velocity oscillations at the wall $\alpha = \tan^{-1} ba^{-1}$. Then, as follows from (25) and (26) $\hat{\theta}_1 = \hat{\theta}_2 = 0$. Hence only weakly damped thermal and transverse waves exist in the medium

$$T = \theta_0 \exp(k_2 x)\cos(\omega t - k_1 x) \quad (30)$$

$$v = V_0 \exp(k_2 x)\cos(\omega t - k_1 x + \tan^{-1} ba^{-1}).$$

The maximum amplitude of temperature and velocity oscillations coincides with the amplitude of oscillations at the bounding surface, and the phase shift between thermal and velocity waves at any point of the wave agrees with that at the wall.

The boundary conditions can be chosen so, that a weakly attenuated wave will be damped

Table 1.

γ deg cm / $\omega/2\pi$ c/s	water									air								
	L (cm)			l (cm)			ξ			L (cm)			l (cm)			ξ		
	0·1	1	10	0·1	1	10	0·1	1	10	0·1	1	10	0·1	1	10	0·1	1	10
10^{-1}	—	—	0·71	—	—	0·49	—	—	1·2	—	11·5	19	—	1·82	1·02	—	6·3	20·7
10^{-2}	2·2	3·9	7·1	1·57	0·88	0·49	1·37	4·4	12	60	115	195	3·3	1·82	1·02	20	63	207
10^{-3}	22	39	71	1·57	0·88	0·49	13·7	44	120	6.10^2	1150	1950	3·3	1·82	1·02	2.10^2	630	2070
10^{-4}	220	390	710	1·57	0·88	0·49	137	440	1200	6.10^3	11500	19500	3·3	1·82	1·02	2.10^3	63.10^2	20700

and all the energy at the boundary will be used for generation of a strongly attentuated wave. This is the case when

$$V_0 = b\theta_0 \sin^{-1} \alpha \qquad \alpha = -\tan^{-1} ba^{-1}.$$

Two more problems are of a great interest: what waves are generated only due to the wall temperature and bounding surface oscillations. It is not difficult to show that in these cases both velocity and temperature oscillations are generated, i.e. the stimulation of transverse oscillations by thermal ones and stimulation of temperature oscillations by transverse oscillations of the the bounding surface in nonisothermal fluid in a gravity field may be assumed. Let the amplitude of the wall temperature oscillations θ_0 be zero, then from equations (23)–(29) it follows

$$T = \frac{V_0}{2a}[\exp k_2 x \cos(\omega t - k_1 x) - \exp \hat{k}_2 x \cos(\omega t - \hat{k}_1 x)] \qquad (31)$$

$$v = V_0(2\cos\psi)^{-1}[\exp(k_2 x)\cos(\omega t - k_1 x + \psi) - \exp k_2 x \cos(\omega t - k_1 x - \psi)],$$
$$\psi = \tan^{-1} ba^{-1}. \qquad (32)$$

Let the wall be at rest now, $v_0 = 0$. Then equations (31) and (32) hold if the wave amplitudes are substituted by $\theta_0(2 \cos \varphi)^{-1}$ and $\theta_0(a^2 + b^2)(2a)^{-1}$, respectively. In both cases weakly and strongly attenuated waves are generated.

So far the oscillations of temperature and transverse velocity component have been discussed. By substituting the expressions obtained for T and v into equations of state (7), continuity (8) and motion (9), it is not difficult to find weakly and strongly attenuated waves of density, transverse velocity components and pressure. The whole complex of density, temperature, velocity, sound and pressure waves existing in a nonisothermal fluid, contained in a gravity field, only due to thermal compressibility will be referred to as thermoconvective. Of the greatest interest are, of course, the weakly attenuated thermal and transverse waves predicted and studied in the present paper.

REFERENCES

1. A. V. Luikov and B. M. Berkovsky, Weakly damped transverse and thermal waves in fluids, *Dokl. Akad. Nauk SSSR* (in press).
2. A. V. Luikov and B. M. Berkovsky, Thermoconvective waves in visco-elastic fluids, *J. Engng Phys. (Inzh. Fiz. Zh)* (in press).
3. A. V. Luikov, *Heat Conduction Theory* (*Teoriya teploprovodnosti*) Vysshaya Shkola, Moscow (1967).
4. A. V. Luikov and Y. A. Mikhailov, *Theory of Heat and Mass Transfer* (*Teoriya teplo-i massoperenosa*). Gosenergoizdat (1963).
5. L. D. Landau and E. M. Livshits, *Solid Mechanics* (*Mekhanika sploshnykh sred*). GITTL (1954).
6. A. M. Tikhonov and A. A. Samarsky, *Equations of* Nauka (1966).
7. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*. Oxford University Press (1948).
8. B. M. Berkovsky, Transverse waves in non-Newtonian fluids. Proceedings of the 3rd All-Union Heat and Mass Transfer Conference, **10**, 339–345 Nauka Teknika, (1968).
9. L. G. Fillipenko, Condition for the absence of convection in real unbounded fluid, *Mekhanika Zhidk. Gaza* No. 4, 125–127 (1967).
10. V. S. Sorokin, Stability of uniformly heated gas in the gravity field, *PMM* **17**, 149–157 (1953).
11. A. V. Luikov and B. M. Berkovsky, Onset of convection in cavities with variable wall temperatures, *Inhz. Fiz. Zh.* **15**, 969–975 (1968).

THERMOCONDUCTIVE WAVES

Abstract—In the paper the possibility of existence of weakly damped transverse, thermal and sound waves in a viscous, heat-conducting, thermally compressible fluid is predicted in the presence of a gravity field. The frequency spectrum, penetration length and wavelengths were analysed of waves propagating with small attenuation. The possibility is also discussed of designing a mechanical generator for thermal waves and stimulating transverse and longitudinal oscillations by a thermal method.

ONDES THERMOCONVECTIVES

Résumé—Dans cet article, la possibilité d'une distribution d'ondes sonores et thermiques transversales faiblement amorties dans un fluide compressible visqueux et conducteur de la chaleur est prédite en présence d'un champ magnétique. Le spectre de fréquence, la longueur de pénétration et les longueurs d'ondes de la distribution ayant une faible atténuation sont analysées. On discute aussi la possibilité de concevoir un générateur mécanique d'ondes thermiques et de stimuler des oscillations transversales et longitudinales par une méthode thermique.

THERMOKONVEKTIVE WELLEN

Zusammenfassung—In der Arbeit wird die Ausbreitung von schwach gedämpften Quer-, thermischen- und Schallwellen in einer zähen, wärmeleitenden, kompressiblen Flüssigkeit in Gegenwart eines Magnetfeldes ermittelt. Das Frequenzspektrum, die Eindringtiefe und die Länge von Wellen, die sich mit schwacher Dämpfung ausbreiten, werden analysiert.

Es wird auch die Möglichkeit zur Herstellung eines Generators für thermische Wellen und zur Erregung von Quer- und Längsschwingungen nach einer thermischen Methode diskutiert.